ELSEVIER
Topology and its Applications 95 (1999) 217–228
TOPOLOGY AND ITS APPLICATIONS
www.elsevier.com/locate/topol

Homology category of 3-manifolds

J.C. Gómez-Larrañaga [a,b,*], F. González-Acuña [c], V.G. Turaev [d]

[a] Department of Mathematics, Florida State University, Tallahassee, FL 32306, USA
[b] CIMAT, A.P. 402, 36000 Guanajuato, Gto., México
[c] Instituto de Matemáticas, U.N.A.M., Ciudad Universitaria, 04510, México, D.F., México
[d] Institut de Recherche Mathématique Avancée, Université Louis Pasteur – C.N.R.S., Paris, France

Received 1 February 1997; received in revised form 21 November 1997

Abstract

The H_1-category of orientable closed 3-manifolds is calculated.

Keywords: Homology category; H_1-category; 3-manifolds; Cohomology ring; Surgery; Homology sphere; Boundary link; Homology boundary link

AMS classification: 57N10

1. Introduction

Let F be a functor from the category *Top* of topological spaces to any category $\mathcal{C}$. Let X be a topological space. We say that a subset U of X is F*-contractible* (in X) if $F(i): F(U) \to F(X)$ is constant [7, Definition 8.2] where $i: U \to X$ is the inclusion. We define $\mathrm{cat}_F(X)$ as the smallest n such that X can be covered by n sets, open and F-contractible in X. If no such integer exists, then $\mathrm{cat}_F(X) = \infty$.

If F is the natural functor from *Top* to the homotopy category, $\mathrm{cat}_F(X)$ is the usual *Lusternik–Schnirelmann category* of X, $\mathrm{cat}(X)$.

If π_1 is the functor from *Top* to the category of groups defined on objects by $\pi_1(X) = *_{x\in X}\pi_1(X,x)$, where $*$ denotes free product, then a subset U of X is π_1-contractible if and only if every loop in U is nullhomotopic in X. The invariant $\mathrm{cat}_{\pi_1}(X)$ was defined by Fox [2], and has been studied in [1,3]. In [5] it was proved that, if M^3 is a closed 3-manifold, $\mathrm{cat}(M^3)$ and $\mathrm{cat}_{\pi_1}(M^3)$ depend only of $\pi_1(M^3)$. More precisely:

Theorem [5]. *Let M^3 be a closed* 3*-manifold.*

* Corresponding author. E-mail: jcarlos@fractal.cimat.mx.

PII: S0166-8641(98)00005-4

(i) *If* $\pi_1(M^3) = 1$ *then* $\mathrm{cat}(M^3) = 2$ *and* $\mathrm{cat}_{\pi_1}(M^3) = 1$.
(ii) *If* $\pi_1(M^3)$ *is free nontrivial then* $\mathrm{cat}(M^3) = 3$ *and* $\mathrm{cat}_{\pi_1}(M^3) = 2$.
(iii) *If* $\pi_1(M^3)$ *is not free then* $\mathrm{cat}(M^3) = \mathrm{cat}_{\pi_1}(M^3) = 4$.

If H is the functor from *Top* to the category Ab of Abelian groups defined on objects by $H(X) = \bigoplus_{i>0} H_i(X; \bigoplus_{m\geqslant 0} Z_m)$ where H_i is ith singular homology, and X is a finite complex then $\mathrm{cat}_H(X)$ is the complete *homological category* of X defined and studied by Fox.

We also consider the functor $H_1 : Top \to Ab$ where H_1 associates to a space its first singular homology group with integral coefficients.

Clearly $\mathrm{cat}_{H_1}(X) \leqslant \mathrm{cat}_H(X) \leqslant \mathrm{cat}(X)$ and $\mathrm{cat}_{H_1}(X) \leqslant \mathrm{cat}_{\pi_1}(X) \leqslant \mathrm{cat}(X)$. Also, if M^n is an n-manifold, $\mathrm{cat}(M^n) \leqslant n + 1$ [8,13] and, if M^n is closed, $\mathrm{cat}_H(M^n) \geqslant 2$. Thus, for a closed orientable 3-manifold M^3 the number $\mathrm{cat}_H(M^3)$ belongs to the set $\{2, 3, 4\}$. By [2], $\mathrm{cat}_H(M^3) = 2$ if and only if M^3 is an integral homology sphere. It remains to find out when $\mathrm{cat}_H(M^3) = 3$. In [4] it was proved that, if M^3 is a closed orientable 3-manifold with $\mathrm{cat}_H(M^3) \leqslant 3$, then the cohomology ring of M^3 with integer coefficients is that of a connected sum of S^3 with n copies of $S^1 \times S^2$ for some $n \geqslant 0$. In the present paper we prove the converse of this result.

Similarly, the number $\mathrm{cat}_{H_1}(M^3)$ belongs to the set $\{1, 2, 3, 4\}$. It is obvious that $\mathrm{cat}_{H_1}(M^3) = 1$ if and only if M^3 is an integral homology sphere. It remains to find out when $\mathrm{cat}_{H_1}(M^3) = 2$ and when $\mathrm{cat}_{H_1}(M^3) = 3$. We give the complete answers to these questions in the following two theorems.

If n is a natural number $n(S^1 \times S^2)$ denotes the connected sum of n copies of $S^1 \times S^2$; if $n = 0$, $n(S^1 \times S^2)$ denotes S^3. The singular cohomology ring of X is denoted by $H^*(X)$. If G is a group, let $G_0 = G$, $G_{n+1} = [G, G_n]$, $G_\omega = \bigcap_{n\geqslant 0} G_n$ and $G' = G_1$. A surface is a connected 2-manifold. We identify S^1 with the space of complex numbers modulus 1.

Theorem (Theorems 1 and 2). *Let* M^3 *be a closed oriented* 3-*manifold. The following statements are equivalent*:
(i) $\mathrm{cat}_{H_1}(M^3) \leqslant 2$.
(ii) G/G_ω *is free where* $G = \pi_1(M^3)$.
(iii) $H_1(M^3) \approx Z^n$ *for some* $n \geqslant 0$ *and there exist disjoint oriented surfaces* $F_1, \ldots, F_n$ *representing a basis of* $H_2(M^3)$.

Theorem (Theorems 1 and 3). *Let* M^3 *be a closed oriented* 3-*manifold. The following statements are equivalent*:
(i) $\mathrm{cat}_{H_1}(M^3) \leqslant 3$.
(ii) $H^*(M^3) \approx H^*(n(S^1 \times S^2))$ *for some* n.
(iii) $H_1(M^3) \approx Z^n$ *for some* $n \geqslant 0$ *and the image under the homomorphism* $H_3(\pi_1 M^3) \to H_3(\pi_1 M^3/(\pi_1 M^3)')$ *of the fundamental class is* 0.
(iv) $H_1(M^3) \approx Z^n$ *for some* $n \geqslant 0$ *and there are oriented surfaces* $F_1, \ldots, F_n$ *representing a basis of* $H_2(M^3)$ *such that* $F_i \cap F_j \cap F_k = \emptyset$ *if* $i < j < k$.
(v) $\mathrm{cat}_H(M^3) \leqslant 3$.

We also give surgical characterizations of the manifolds M^3 with $\mathrm{cat}_{H_1}(M^3) = m$.

An n-component link $k_1 \cup \cdots \cup k_n$ in a homology sphere Σ^3 will be called a *boundary link* if there exist disjoint compact orientable surfaces $S_1, \ldots, S_n \subset \Sigma^3$ with $\partial S_i = k_i$. (In the usual definition [14], $\Sigma^3 = S^3$.)

An n-component link $L = k_1 \cup \cdots \cup k_n$ in a homology sphere Σ^3 will be called a *homology boundary link* if there exist n disjoint compact oriented surfaces $F_1, \ldots, F_n$ in the exterior E of L such that each component of each ∂F_i is a longitude and $[\partial F_i] = [\lambda_i] \in H_1(\partial E)$ where λ_i is a longitude of k_i. An equivalent definition [14] is: (Σ^3, L) is a homology boundary link if G/G_ω is free where $G = \pi_1(\Sigma^3 - L)$.

Let $M(L)$ be the manifold obtained by longitudinal surgery on a link L in a homology sphere. The closed orientable manifolds M^3 with $H_1(M^3) = Z^n$ are those of the form $M(L)$ where L is an n-component link with all linking numbers equal to zero (Proposition 3).

We prove that the manifolds M^3 with $\mathrm{cat}_{H_1}(M^3) = 2$ are those obtained by longitudinal surgery on a boundary link (Theorem 4). Also (Theorem 5), if L is a link in a homology sphere with all linking numbers equal to zero, $\mathrm{cat}_{H_1} M(L) = 2$ iff L is a homology boundary link, and $1 < \mathrm{cat}_{H_1} M(L) \leqslant 3$ iff all Milnor μ-invariants $\mu(i, j, k)$ [10,18] of weight three of L vanish. We will be working in the PL category.

2. Homotopic and geometric characterizations

In this section we prove that, if $m = 2$ or 3, the condition $\mathrm{cat}_{H_1}(M^3) \leqslant m$ is equivalent to the condition that the canonical map from M^3 into the classifying space $BH_1(M^3)$ being deformable to the $(m-1)$-skeleton of $BH_1(M^3)$. More geometrically, this condition is also equivalent to the existence of n surfaces representing a basis of the free abelian group $H_2(M^3)$ such that the intersection of any m of them is empty.

Lemma 1. *Let G be a group and let F be a free group. The following statements are equivalent*:

(i) *There is a homomorphism from G to F inducing an isomorphism from G/G' onto F/F'.*

(ii) *There is an epimorphism from G to F inducing an isomorphism from G/G' onto F/F'.*

(iii) $G/G_\omega \approx F$.

Proof. First we show that (i) implies (ii). Let $\alpha : G \to F$ be a homomorphism inducing an isomorphism from G/G' onto F/F'.

Consider the commutative diagram

$$\begin{tikzcd}
G \arrow[r, "\alpha"] \arrow[d] & \alpha(G) \arrow[r, "i"] \arrow[d, "\rho"'] & F \arrow[d] \\
G/G' \arrow[r, "\hat{\alpha}"] & \alpha(G)/\alpha(G)' \arrow[r, "\hat{\imath}"] & F/F'
\end{tikzcd}$$

in which i is inclusion and $\hat{\alpha}$ (respectively $\hat{\imath}$) is induced by α (respectively i). Since α and ρ are epimorphisms so is $\hat{\alpha}$. Since $\hat{\imath}\hat{\alpha}$ is an isomorphism $\hat{\alpha}$ is injective. Hence $\hat{\alpha}$ and $\hat{\imath}$ are isomorphisms and so $\operatorname{rank}\alpha(G) = \operatorname{rank} F$. Hence, there is an isomorphism ψ from $\alpha(G)$ onto F. Then the composition $G \xrightarrow{\alpha} \alpha(G) \xrightarrow{\psi} F$ is the required epimorphism.

Obviously (ii) implies (i). That (ii) implies (iii) is a consequence of [15, Theorem 3.4] and the fact that $F_\omega = 1$ [9].

Finally, we will show that (iii) implies (ii). There is an epimorphism $\varphi : G \to F$ with kernel G_ω. Since $G_\omega \subset G'$, φ induces an isomorphism from G/G' onto F/F'. □

We will need the following two propositions.

Proposition 1 [5, §2]. *Let M be a paracompact, locally pathwise connected space and let n be a natural number. In order that $\operatorname{cat}_{H_1}(M) \leqslant n$ it is necessary and sufficient that there exist a complex L of dimension less than n and a map $f : M \to L$ such that $f_* : H_1(M) \to H_1(L)$ is an isomorphism.*

Proposition 2 [19, Lemma 2.1]. *Let L be a connected complex, M^3 a closed oriented 3-manifold and $f : M^3 \to L$ a mapping inducing an isomorphism $H_1(M^3) \to H_1(L)$, $\nu = f_*([M^3]) \in H_3(L)$ where $[M^3] \in H_3(M^3)$ is the fundamental class. Then the homomorphism $\operatorname{Torsion} H^2(L) \xrightarrow{\frown \nu} \operatorname{Torsion} H_1(L)$ is an isomorphism. In particular, if $\nu = 0$ then $\operatorname{Torsion} H_1(M^3) = 0$.*

The following lemma is essentially proved in [4]; we will, however, give here a different proof.

Lemma 2. *Let M^3 be a closed orientable 3-manifold. If $\operatorname{cat}_{H_1}(M^3) \leqslant 3$ then $H_1(M^3)$ is free abelian.*

Proof. By Proposition 1 there is a map f from M^3 to a 2-complex L^2 such that $f_* : H_1(M^3) \to H_1(L^2)$ is an isomorphism. Clearly $f_* : H_3(M^3) \to H_3(L^2)$ is trivial. Then Proposition 2 implies that $\operatorname{Torsion} H_1(M^3) = 0$. □

If G is a group, the classifying space BG of G is a complex with fundamental group G and trivial higher homotopy groups. Thus if $H_1(M^3) \approx Z^n$ and $n \geqslant 1$, one may take an n-torus $S^1 \times S^1 \times \cdots \times S^1$ as $BH_1(M^3)$. A map into BG is *m-deformable* if it is homotopic to a map whose image is contained in the m-skeleton of BG.

Theorem 1. *Let M^3 be an orientable closed 3-manifold and let $m = 2$ or 3. Then the following properties are equivalent*:

(i) $\operatorname{cat}_{H_1}(M^3) \leqslant m$.
(ii) *A map $g : M^3 \to BH_1(M^3)$, with $g_* : H_1(M^3) \to H_1(BH_1M^3)$ an isomorphism, is $(m-1)$-deformable.*

(iii) $H_1(M^3) \approx Z^n$, *and there exist oriented* 2*-submanifolds, in general position,* $F_1, \ldots, F_n$ *of* M^3 *representing a basis of* $H_2(M^3)$ *where the intersection of* m *different* F_i*'s is empty.*

(iv) $H_1(M^3) \approx Z^n$*, and there exist oriented surfaces, in general position,* $F_1, \ldots, F_n$ *of* M^3 *representing a basis of* $H_2(M^3)$ *where the intersection of* m *different* F_i*'s is empty.*

Remark. Property (ii) is equivalent to: "there exists $g: M^3 \to BH_1(M^3)$, with

$$g_*: H_1(M^3) \to H_1(BH_1M^3)$$

being an isomorphism, whose image is contained in the $(m-1)$-skeleton of $BH_1(M^3)$".

Proof. First we show that (i) implies (ii). By Proposition 1 there is a map $f: M^3 \to L^{m-1}$ with L^{m-1} an $(m-1)$-complex, such that

$$f_*: H_1(M^3) \to H_1(L^{m-1})$$

is an isomorphism. Let $K = BH_1(M^3)$ be the classifying space of the group $H_1(M^3)$. Let $h: L^{m-1} \to K$ be a map, with $h_*: H_1(L^{m-1}) \to H_1(K)$ an isomorphism, whose image is contained in the $(m-1)$-skeleton $K^{(m-1)}$ of K. Then $g = h \circ f$ induces an isomorphism of first homology and its image is contained in $K^{(m-1)}$. This proves (ii).

To show that (ii) implies (iii) let $g: M^3 \to BH_1(M^3)$ be an $(m-1)$-deformable map inducing an isomorphism of first homology. Then $g_*: H_3(M^3) \to H_3(BH_1M^3)$ is trivial and so, by Proposition 2, Torsion $H_1(M^3) = 0$. Thus, we have $H_1(M^3) \approx Z^n$ for some n. We may assume $n \geqslant m$. Write $BH_1(M^3) = S^1 \times \cdots \times S^1$ (n factors). We take as the $(m-1)$-skeleton of $BH_1(M^3)$ the set Y of points $(x_1, \ldots, x_n)$ in $BH_1(M^3)$ such that at least $n-m+1$ of its components are 1.

Deform g such that its image is contained in Y and then further deform it so that its component g_i has -1 as a regular value ($i = 1, \ldots, n$) and the image of $(g_1, \ldots, g_n): M^3 \to BH_1(M^3)$ is contained in a neighborhood $N(Y)$ where $N(Y)$ is the set of points $(x_1, \ldots, x_n)$ in $BH_1(M^3)$ such that at most $m-1$ of its components are equal to -1. Define $F_i = g_i^{-1}(-1)$ ($i = 1, \ldots, n$). Then the intersection of m different F_i's is empty.

Since S^1 is oriented we can identify $H_1(S^1)$ with Z. Then we have $\varphi: H_1(S^1 \times \cdots \times S^1) \to Z^n$ defined by $\varphi(\alpha) = (p_{1_*}\alpha, \ldots, p_{n_*}\alpha)$ where $p_i: S^1 \times \cdots \times S^1 \to S^1$ is projection onto the ith factor. The composition φg_* is an isomorphism sending $\alpha \in H_1(M^3)$ to $(\alpha \bullet [F_1], \ldots, \alpha \bullet [F_n])$ where $\bullet$ denotes intersection number. If $\alpha_i \in H_1(M^3)$ is such that $\varphi g_*(\alpha_i) = (0, \ldots, 1, 0, \ldots, 0)$, where the 1 is the ith component, then $\{\alpha_1, \ldots, \alpha_n\}$ is a basis of $H_1(M^3)$ and $\alpha_i \bullet [F_j] = \delta_{ij}$. This implies, by Poincaré duality, that $\{[F_1], \ldots, [F_n]\}$ is a basis of $H_2(M^3)$. This proves (iii).

To show that (iii) implies (i), let M_j ($j = 0, \ldots, m-1$) be the set of points of M^3 that belong to exactly j of the 2-submanifolds $F_1, \ldots, F_n$. Any loop in M_j ($j = 0, \ldots, m-1$) is homotopic, in M, to a loop in M_0 and so, it has zero intersection number with each F_i. This implies that $H_1(M_j) \to H_1(M^3)$ is trivial ($j = 0, \ldots, m-1$) because $H_1(M^3)$ is free abelian. Let U_j be an open neighborhood of M_j that deformation-retracts to M_j. Then $\{U_0, \ldots, U_{m-1}\}$ is a cover of M^3 with H_1-contractible open sets. This proves (i).

Obviously (iv) implies (iii).

Now, we will show that (iii) implies (iv) for $m = 3$. Among all collections of oriented 2-manifolds in general position $F_1, \ldots, F_n$ representing a basis of $H_2(M^3)$ and satisfying $F_i \cap F_j \cap F_k = \emptyset$ for $i < j < k$ choose one such that $\sum_{i=1}^{n} |F_i|$ is minimal, where $|X|$ denotes the number of components of X. We claim that $|F_i| = 1$ for $i = 1, \ldots, n$.

Orient M^3. Then the orientations of M^3 and of each F_i determine, for every component C of F_i, a preferred normal based at a point of C. Now associate to each F_i an oriented graph $\mathcal{F}_i$ as follows. The vertices $V_1, \ldots, V_r$ of $\mathcal{F}_i$ are in one-to-one correspondence with the components of $M^3 - F_i$ and the edges $e_1, \ldots, e_s$ are in one-to-one correspondence with the components of F_i. The terminus (respectively origin) of e_j is V_k if the preferred normal (respectively the negative of the preferred normal) based at a point of the component of F_i corresponding to e_j points to the component of $M^3 - F_i$ corresponding to V_k. Since M^3 is connected so is $\mathcal{F}_i$.

If some $\mathcal{F}_i$ had a terminal vertex, that is, a vertex which is an extremity of exactly one edge e and e is not a loop, then we could replace F_i by $F_i - C$ where C is the component corresponding to e, contradicting the minimality of $\sum_{i=1}^{n} |F_i|$. Hence no $\mathcal{F}_i$ has a terminal vertex.

Suppose some $\mathcal{F}_i$ had two different edges with the same terminus. Then one could join two different components of F_i with a tube missing $\bigcup_{j \neq k}(F_j \cap F_k)$ to produce an oriented 2-manifold homologous to F_i with less components than F_i, contradicting the minimality condition. Hence no $\mathcal{F}_i$ has two different edges with the same terminus.

Similarly one proves that no $\mathcal{F}_i$ has two edges with the same origin.

Since $\mathcal{F}_i$ does not have two edges with the same terminus or origin and $\mathcal{F}_i$ has no terminal vertex it follows that $\mathcal{F}_i$ is an oriented cycle with n edges. Moreover, we must have $n = 1$ since the n edges represent homologous surfaces and $[F_i] \in H_2(M^3)$ is primitive. Therefore each $\mathcal{F}_i$ is a loop and so each F_i is connected. This proves that (iii) implies (iv) for $m = 3$.

Next we will show that (ii) implies (iv) for $m = 2$. There is a map from M^3 to the 1-skeleton of $BH_1(M^3)$ inducing, on fundamental groups, a homomorphism $\varphi : \pi_1(M^3) \to F$, with F free, which in turn induces an isomorphism from $\pi_1 M^3/(\pi_1 M^3)'$ onto F/F'. Using the implication (i) $\Rightarrow$ (ii) of Lemma 1, we obtain a map f from M^3 to a wedge of n circles $\bigvee^n S^1$ such that $f_* : \pi_1(M^3) \to \pi_1(\bigvee^n S^1)$ is an epimorphism and $f_* : H_1(M^3) \to H_1(\bigvee^n S^1)$ is an isomorphism. Take n points $x_1, \ldots, x_n$, different from the wedge point, one on each circle. We may also assume that f is transverse to each x_i. Among all maps f with the properties above choose one such that $\sum_{i=1}^{n} |f^{-1}(x_i)|$ is minimal. We claim that $|f^{-1}(x_i)| = 1$ $(i = 1, \ldots, n)$. We have $|f^{-1}(x_i)| \geqslant 1$ for each i, since $f_* : \pi_1(M^3) \to \pi_1(\bigvee^n S^1)$ is an epimorphism.

If $|f^{-1}(x_i)| > 1$ for some i then, using Stallings method of arc chasing (see, for example, [6, p. 67]), we can find f_1, homotopic to f and transverse to each x_i, such that $\sum_{i=1}^{n} \left|f_1^{-1}(x_i)\right| < \sum_{i=1}^{n} \left|f^{-1}(x_i)\right|$ contradicting our minimality assumption. Hence $|f^{-1}(x_i)| = 1$ for each i. Thus, defining F_i to be $f^{-1}(x_i)$, we have that each F_i is connected. As in the last part of the proof of the implication (ii) $\Rightarrow$ (iii) one sees that $[F_1], \ldots, [F_n]$ is a basis of $H_2(M^3)$. This proves that (ii) $\Rightarrow$ (iv) for $m = 2$.

We have then completed the proof that (i)–(iv) are equivalent. $\square$

3. Cohomological and π_1 characterizations

Condition (iv) of Theorem 1 gives a geometric characterization of the closed orientable 3-manifolds M^3 satisfying $\mathrm{cat}_{H_1}(M^3) \leqslant m$. In Theorems 2 and 3 we give characterizations in terms of $\pi_1(M^3)$ and cohomological characterizations of the condition $\mathrm{cat}_{H_1}(M^3) \leqslant m$. If we orient M^3, we denote by μ (respectively θ) the image of the fundamental class $[M^3] \in H_3(M^3)$ under the canonical homomorphism $H_3(M^3) \to H_3(H_1 M^3)$ (respectively $H_3(M^3) \to H_3(\pi_1 M^3)$). The condition $\mu = 0$ (respectively $\theta = 0$) is, of course, independent of the orientation given to M^3. The condition $\mathrm{cat}_{H_1}(M^3) \leqslant 3$, which turns out to be equivalent to $\mathrm{cat}_H(M^3) \leqslant 3$, is equivalent (Theorem 3(v)) to $\mu = 0$, that is, using the terminology of [19], to M^3 having the abelian homotopy type of $n(S^1 \times S^2)$ for some $n \geqslant 0$. This can be compared with the fact, implicit in [5], that $\mathrm{cat}(M^3) \leqslant 3$ (or $\mathrm{cat}_{\pi_1}(M^3) \leqslant 3$) is equivalent to $\theta = 0$, that is, to M^3 having the homotopy type of $n(S^1 \times S^2)$ for some $n \geqslant 0$ [16,17]. Thus μ (respectively θ) is the obstruction to covering M^3 with three open sets H-contractible (respectively contractible) in M. In Theorem 3, (ii) $\Rightarrow$ (iii) is essentially proved in [4]. However, we will not use [4] and give a different proof here.

Theorem 2. *Let M^3 be an orientable closed 3-manifold. The following properties are equivalent*:

(i) $\mathrm{cat}_{H_1}(M^3) \leqslant 2$.
(ii) *G/G_ω is free where $G = \pi_1(M^3)$.*

Proof. By Lemma 1, (ii) is equivalent to the existence of a homomorphism from G into a free group F inducing an isomorphism from G/G' onto F/F'. By Proposition 1, (i) is equivalent to the existence of a map f from M^3 into a 1-complex L such that $f_*: H_1(M^3) \to H_1(L)$ is an isomorphism; such a map induces a homomorphism from $\pi_1(M^3)$ into the free group $\pi_1(L)$ inducing an isomorphism in first homology. Also writing $F = \pi_1(L)$ where L is a 1-complex, any homomorphism from G into F inducing an isomorphism on abelianizations is induced by a map $f: M^3 \to L$ such that $f_*: H_1(M^3) \to H_1(L)$ is an isomorphism. Hence (i) and (ii) are equivalent. $\square$

Theorem 3. *Let M^3 be a closed orientable 3-manifold. The following properties are equivalent*:

(i) $\mathrm{cat}_{H_1}(M^3) \leqslant 3$.
(ii) $\mathrm{cat}_H(M^3) \leqslant 3$.
(iii) *The cohomology ring of M^3 is isomorphic to that of $n(S^1 \times S^2)$ for some $n \geqslant 0$.*
(iv) *The homomorphism $H_3(\pi_1 M^3) \to H_3(H_1 M^3)$, induced by abelianization, is trivial.*
(v) $\mu = 0$.

Proof. (i) $\Rightarrow$ (ii) By the arguments of [12] (see also [11]) there exist handlebodies with T_1, T_2 and T_3 such that $T_1 \cup T_2 \cup T_3 = M^3$ and $H_1(T_i) \to H_1(M^3)$ is trivial ($i = 1, 2, 3$). Letting U_i be an open regular neighborhood of T_i we have an open cover $\{U_1, U_2, U_3\}$ of M^3 with 3 open sets H-contractible in M^3.

(ii) $\Rightarrow$ (i) is obvious.

(i) $\Rightarrow$ (v) By Theorem 1 the canonical map $f : M^3 \to BH_1(M^3)$ is 2-deformable and therefore $f_* : H_3(M^3) \to H_3(BH_1M^3)$ is trivial and so the image μ of a generator of $H_3(M^3)$ under f_* is zero.

(v) $\Rightarrow$ (iii) Since the canonical map $f : M^3 \to BH_1(M^3)$ induces an isomorphism on first homology and $\mu = 0$ it follows from Proposition 2 that Torsion $H_1(M^3) = 0$.

Now, if $a, b, c \in H^1(BH_1M^3)$ and $[M^3] \in H_3(M^3)$ is a generator, we have $\langle f^*a \cup f^*b \cup f^*c, [M^3]\rangle = \langle a \cup b \cup c, f_*([M^3])\rangle = \langle a \cup b \cup c, \mu\rangle = 0$ and so $\alpha \cup \beta \cup \gamma = 0$ for any $\alpha, \beta, \gamma \in H^1(M^3)$ since $f^* : H^1(H_1M^3) \to H^1(M^3)$ is an epimorphism.

Since $H_1(M^3)$ is free abelian we can take dual bases $\{e_i^1\}$ in $H^1(M^3)$ and $\{e_j^2\}$ in $H^2(M^3)$. For any i, j, $e_i^1 \cup e_j^1 = 0$ because if we write $e_i^1 \cup e_j^1 = \sum_k n_k e_k^2$, then $0 = e_k^1 \cup e_i^1 \cup e_j^1 = n_k\gamma$, where $\gamma \in H^3(M^3)$ is a generator, so that $n_k = 0$ for $k = 1, \dots, n$. This proves the cohomology ring of M is isomorphic to that of $n(S^1 \times S^2)$ for some $n \geqslant 0$.

(v) $\Rightarrow$ (iv) Let $M^3 = M_1^3 \# \cdots \# M_r^3$ be the connected sum decomposition of M^3 into prime factors. This induces a decomposition of $\pi_1(M^3)$ as a free product, $\pi_1(M_1^3) * \cdots * \pi_1(M_r^3)$, and, abelianizing, a decomposition of $H_1(M^3)$ as a direct sum $H_1(M_1^3) \oplus \cdots \oplus H_1(M_r^3)$. We have commutative diagrams

$$\begin{tikzcd} H_3(\pi_1 M^3) \arrow[r, "\alpha"] & H_3(H_1 M^3) \arrow[d, shift left, "p_k"] \\ H_3(\pi_1 M_k^3) \arrow[u, "j_k"] \arrow[r, "\alpha_k"] & H_3(H_1 M_k^3) \arrow[u, shift left, "i_k"] \end{tikzcd},$$

where j_k and i_k are induced by inclusions, p_k by projection, α and α_k by abelianization. Notice that $p_l i_k$ is trivial if $l \neq k$ and the identity if $l = k$.

Giving an orientation to M^3, denote by θ (respectively by μ) the image of the fundamental class in $H_3(\pi_1 M^3)$ (respectively in $H_3(H_1M^3)$). We have $\theta = \sum_{k=1}^r j_k(\theta_k)$ where θ_k generates $H_3(\pi_1 M_k^3)$. The elements $j_1(\theta_1), \dots, j_r(\theta_r)$ generate $H_3(\pi_1 M^3)$. Also, we have

$$0 = \mu = \alpha(\theta) = \sum_{k=1}^r \alpha j_k(\theta_k) = \sum_{k=1}^r i_k \alpha_k(\theta_k).$$

Hence, for any l,

$$0 = p_l\left(\sum_{k=1}^r i_k\alpha_k(\theta_k)\right) = \alpha_l(\theta_l)$$

and so

$$\alpha j_l(\theta_l) = i_l\alpha_l(\theta_l) = 0.$$

Therefore $\alpha = 0$.

(iv) $\Rightarrow$ (v) is obvious.

(iii) $\Rightarrow$ (i) We will prove that (ii) in Theorem 1 holds for $m = 3$. Identify $BH_1(M^3)$ with the n-torus $T^n = S^1 \times S^1 \times \cdots \times S^1$ where n is the rank of $H_1(M^3)$, which may be assumed to be $n \geqslant 3$.

Write $T_{ij} = A_1 \times A_2 \times \cdots \times A_n$ where

$$A_k = \begin{cases} \{1\} & \text{if } k \notin \{i, j\}, \\ S^1 & \text{if } k \in \{i, j\}, \end{cases}$$

and $T_{ijk} = A_1 \times A_2 \times \cdots \times A_n$ where

$$A_l = \begin{cases} \{1\} & \text{if } l \notin \{i, j, k\}, \\ S^1 & \text{if } l \in \{i, j, k\}. \end{cases}$$

Then $T^{(2)} = \bigcup_{i<j} T_{ij}$ is the 2-skeleton of T^n and $T^{(3)} = \bigcup_{i<j<k} T_{ijk}$ is the 3-skeleton of T^n.

We can assume that the image of a canonical map $f : M^3 \to T^n$ is contained in $T^{(3)}$. For $i < j < k$ take a 3-ball B_{ijk} contained in $T_{ijk} - T^{(2)}$. We can assume that every component of $f^{-1}(B_{ijk})$ is a 3-ball that is mapped homeomorphically onto B_{ijk}. Write $W = M^3 - \bigcup_{i<j<k} f^{-1}(\mathring{B}_{ijk})$. Orient M^3; each T_{ijk} has a natural orientation. If C is a component of $f^{-1}(B_{ijk})$, let ε_C be 1 (respectively (-1)) if $f|C : C \to f(C)$ preserves (respectively reverses) orientation. Let $p_{ijk} : T^n \to T_{ijk}$ be the projection.

Denoting the fundamental class of $H^3(T_{ijk})$ by $a \cup b \cup c$, where $a, b, c \in H^1(T_{ijk})$, we have $(p_{ijk} \circ f)^*(a \cup b \cup c) = (p_{ijk} \circ f)^*(a) \cup (p_{ijk} \circ f)^*(b) \cup (p_{ijk} \circ f)^*(c) = 0$, that is, $\deg(p_{ijk} \circ f) = 0$ and so, for $i < j < k$, $\sum \varepsilon_C = 0$ where C runs over the components of $f^{-1}(B_{ijk})$.

Now we use Stallings' arc-chasing method. Fix i_0, j_0 and k_0 with $i_0 < j_0 < k_0$. Take components C_1 and C_2 of $f^{-1}(B_{i_0 j_0 k_0})$ such that $\varepsilon_{C_1} = 1$ and $\varepsilon_{C_2} = -1$. Let $q_1 \in \partial C_1$, $q_2 \in \partial C_2$ be such that $f(q_1)$ and $f(q_2)$ are the same point q. Let α be an oriented arc in W, from q_1 to q_2.

Since $f_* : \pi_1(M^3) \to \pi_1(T^{(3)})$ is an epimorphism, there is a loop β in W, based at q_1, such that $f\beta$ and $f\alpha$ represent the same element of $\pi_1(T^{(3)}, q)$. Let γ be an arc in W from q_1 to q_2 that is homotopic to $\beta^{-1}\alpha$. Then $f\gamma$ represents the trivial element of $\pi_1(T^{(3)} - \bigcup_{i<j<k} \mathring{B}_{ijk})$. Using the homotopy extension property one obtains a map $f_1 : M^3 \to T^{(3)}$ such that $f_1^{-1}(\bigcup B_{ijk}) = f^{-1}(\bigcup B_{ijk})$, $f_1|f^{-1}(\bigcup B_{ijk}) = f|f^{-1}(\bigcup B_{ijk})$ and f_1 maps γ to the point q. Furthermore, we can assume that there is a tubular neighborhood N of γ in W such that $C_1 \cup N \cup C_2$ is a 3-ball B and $f_1(N) \subset \partial B_{i_0 j_0 k_0}$. Since $f_1|\partial B : \partial B \to \partial B_{i_0 j_0 k_0}$ has degree 0 there is a map $f_2 : M^3 \to T^{(3)}$ such that $f_2|M^3 - \mathring{B} = f_1|M^3 - \mathring{B}$ and $f_2|(\mathring{B}) \subset \partial B_{i_0 j_0 k_0}$. This map is homotopic to f and $f_2^{-1}(\bigcup \mathring{B}_{ijk}) = f^{-1}(\bigcup \mathring{B}_{ijk}) - \mathring{C}_1 - \mathring{C}_2$. Repeating this construction we eventually obtain a map, homotopic to f, whose image is contained in $T^{(3)} - \bigcup \mathring{B}_{ijk}$. Composing this map with a deformation of $T^{(3)} - \bigcup \mathring{B}_{ijk}$ into $T^{(2)}$ we see that (ii) of Theorem 1 holds and, therefore, $\operatorname{cat}_{H_1}(M^3) \leqslant 3$. $\square$

4. Surgical characterizations

In this section we give, for $m \in N$, surgical characterizations of the 3-manifolds M^3 such that $\mathrm{cat}_{H_1}(M^3) = m$.

An n-component link L in a homology sphere Σ^3 is a *boundary link* if there exist n disjoint compact orientable surfaces with nonempty boundary whose union has L as its boundary. (Compare [14].)

A link L in a homology sphere Σ^3 with components $k_1, \ldots, k_n$, is a *homology boundary link* if there exist n disjoint compact oriented surfaces $F_1, \ldots, F_n$ in the exterior E of L such that each component of each ∂F_i is a longitude and $[\partial F_i] = [\lambda_i] \in H_1(\partial E)$ where λ_i is a longitude of k_i. An equivalent definition [14] is: (Σ^3, L) is a homology boundary link if G/G_ω is free where $G = \pi_1(\Sigma^3 - L)$.

We will see that the manifolds M^3 with $\mathrm{cat}_{H_1}(M^3) = 2$ are the ones obtained by 0-surgery on boundary links, and that a manifold M^3 obtained by 0-surgery on a link (Σ^3, L) with vanishing linking numbers has H_1-category two iff (Σ^3, L) is a homology boundary link.

Proposition 3. *Let M^3 be a closed orientable 3-manifold and $n \in N$. Then $H_1(M^3) \approx Z^n$ if and only if M^3 can be obtained by longitudinal surgery on an n-component link L in a homology sphere Σ^3 all of whose linking numbers are zero.*

Proof. If M^3 is obtained by integral surgery on an oriented n-component link $L_1 \cup \cdots \cup L_n$ in an oriented homology 3-sphere then $H_1(M^3) \approx Z^n / \langle (l_{i1}, l_{i2}, \ldots, l_{in})\colon\ 1 \leqslant i \leqslant n \rangle$ where l_{ii} is the (integral) surgery coefficient of L_i and $l_{ij} = lk(L_i, L_j)$ if $i \neq j$. Thus, $H_1(M^3) \approx Z^n$ if and only if all l_{ij} are 0. This proves sufficiency.

Let K be an oriented link in M^3 with components $K_1, \ldots, K_n$ which represent a basis of $H_1(M^3)$. Let K_i^+ be a framing of K_i, that is, a circle such that $K_i \cup K_i^+$ bounds an annulus in M^3. We can assume that K_i^+ is contained in the boundary of a tubular neighborhood T_i of K_i. Denote $M^3 - \mathrm{int}(T_1 \cup \cdots \cup T_n)$ by E. Now, perform surgery on K so as to kill $K_1^+, \ldots, K_n^+$ (that is, in the disjoint union $(S^1 \times D^2)_1 \amalg \cdots \amalg (S^1 \times D^2)_n \amalg E$ identify $x \in \partial(S^1 \times D^2)_i$ with $\varphi_i(x) \in \partial T_i$ where $\varphi_i : \partial(S^1 \times D^2)_i \to \partial T_i$ is a homeomorphism mapping $(1 \times \partial D^2)_i$ onto K_i^+). Denote the resulting closed 3-manifold by Σ^3 and $\bigcup_{i=1}^n (S^1 \times \{0\})_i$ by L (an n-component link in Σ^3). Notice that the inverse surgery on (Σ^3, L) produces M^3. Now $H_1(E) \approx Z^n$ (see, for example, [4, Lemma 1]) and, therefore, the inclusion induced epimorphism $H_1(E) \to H_1(M^3)$ is an isomorphism. Since $K_1^+, \ldots, K_n^+$ represent a basis of $H_1(M)$ they also represent a basis of $H_1(E)$. Hence

$$H_1(\Sigma^3) \approx \frac{H_1 E}{\langle [K_1^+], \ldots, [K_n^+] \rangle} = 0$$

and Σ^3 is a homology sphere. By the argument at the beginning of the proof, all the linking numbers of L are 0 and the n surgery coefficients of L are 0. $\square$

Theorem 4. *Let M^3 be a closed orientable 3-manifold. Then $\mathrm{cat}_{H_1}(M^3) = 2$ if and only if M^3 can be obtained by 0-surgery on a boundary link.*

Theorem 5. *Let M^3 be the manifold obtained by* 0*-surgery on a link L in a homology sphere Σ^3 such that all linking numbers are* 0*. Then*

(i) $1 < \operatorname{cat}_{H_1}(M^3) \leqslant 3$ *if and only if the Milnor invariants $\mu(i, j, k)(L)$ are zero.*

(ii) $\operatorname{cat}_{H_1}(M^3) = 2$ *if and only if (Σ^3, L) is a homology boundary link.*

Proof of Theorems 4 and 5. We will prove first that 0-surgery on a homology boundary link (Σ^3, L) produces a manifold M^3 with $\operatorname{cat}_{H_1}(M^3) = 2$. Let $k_1, \ldots, k_n$ be the components of L and let $F_1, \ldots, F_n$ be disjoint compact oriented surfaces in the exterior E of the link (Σ^3, L) such that each component of each ∂F_i is a longitude and $[\partial F_i] = [\lambda_i] \in H_1(\partial E)$ where λ_i is a longitude of k_i. Capping off the surfaces $F_1, \ldots, F_n$ with disjoint 2-disks contained in the surgery tori we obtain disjoint orientable surfaces $\widehat{F}_1, \ldots, \widehat{F}_n$ in M^3. Notice that, if m_i is a suitably oriented meridian of k_i contained in ∂E, then $m_i \bullet \widehat{F}_j = \delta_{ij}$ and $m_1, \ldots, m_n$ represent a basis of $H_1(E)$ and, therefore, of $H_1(M^3)$. Thus $\{[\widehat{F}_1], \ldots, [\widehat{F}_n]\}$ is the basis of $H_2(M^3)$ dual of the basis $\{[m_1], \ldots, [m_n]\}$ of $H_1(M^3)$. By Theorem 1, $\operatorname{cat}_{H_1} M^3 = 2$. Therefore longitudinal surgery on a homology boundary link produces a manifold with $\operatorname{cat}_{H_1} = 2$. In particular, this proves sufficiency in Theorem 4 since a boundary link is a homology boundary link.

To prove necessity in Theorem 4 assume $\operatorname{cat}_{H_1}(M^3) = 2$. Then by Theorem 1(iv), $H_1(M^3) \approx Z^n$, for some $n \in N$, and there exist disjoint oriented closed surfaces $F_1, \ldots, F_n$ in M^3 such that $[F_1], \ldots, [F_n]$ is a basis of $H_2(M^3)$.

The complement of $\bigcup_{i=1}^n F_i$ in M^3 is connected. Hence, for any i with $1 \leqslant i \leqslant n$, there is an oriented simple closed curve α_i intersecting F_i at exactly one point and transversely with $\alpha_i \cap F_j = \emptyset$ for $j \neq i$. The classes $[\alpha_1], \ldots, [\alpha_n]$ form a basis of $H_1(M^3)$. Let Σ^3 be a 3-manifold obtained by surgery on $\alpha_1 \cup \cdots \cup \alpha_n$ and let L_i be the simple closed curve in Σ^3 that replaces α_i. As shown in the proof of Proposition 3, Σ^3 is a homology sphere. Now, for $i = 1, \ldots, n$, F_i minus a disk neighborhood of $\alpha_i \cap F_i$ is a surface in $\Sigma^3 - \bigcup_{i=1}^n L_i$ whose boundary is a longitude of L_i. These n surfaces with boundary are disjoint. Hence (Σ^3, L) is a boundary link.

To prove necessity in Theorem 5(ii), let (M^3, α) be obtained by longitudinal surgery on an n-component link L in a homology sphere Σ^3 such that all linking numbers of L are 0. Assume $\operatorname{cat}_{H_1}(M^3) = 2$. Notice that $H_1(M^3) \approx Z^n$ and $H_1(M^3 - L) \approx Z^n$. Then by Theorem 4 and Lemma 1, there is an epimorphism $\pi_1(M^3) \xrightarrow{\varphi} F_n$ inducing an isomorphism $\pi_1 M^3/(\pi_1 M^3)' \xrightarrow{\varphi_*} F_n/F_n'$. Since $\Sigma^3 - L \approx M^3 - \alpha$ we also have an epimorphism $\pi_1(\Sigma^3 - L) \to \pi_1(M^3)$ inducing an isomorphism $H_1(\Sigma^3 - L) \to H_1(M^3)$. Hence there is an epimorphism $\pi_1(\Sigma^3 - L) \to F_n$ inducing an isomorphism on abelianizations and therefore (Σ^3, L) is a homology boundary link.

We now prove Theorem 5(i). Represent a basis for $H_1(E)$ by meridians, E being the exterior of L in Σ^3. We have an isomorphism $H_1(E) \to H_1(M^3)$ (see the proof of Proposition 3) and therefore these meridians represent a basis $\{x_1, \ldots, x_n\}$ for $H_1(M^3)$. Let $\omega(1), \ldots, \omega(n)$ be the basis of $H^1(M^3)$ such that $\langle \omega(i), x_j \rangle = \delta_{ij}$. Denote by $[M^3]$ a generator of $H_3(M^3)$.

By [18, Theorem $\mathcal{C}$] (see also the lines preceding 1.5 in [18]) one has $-\omega(i_1) \cup \omega(i_2) \cup \omega(i)([M^3]) = \mu(i_1, i_2, i)$.

Hence $\mu(i, j, k) = 0$ for any indices i, j, k in $\{1, \ldots, n\}$ if and only if $\omega(i) \cup \omega(j) \cup \omega(k) = 0$ for any such indices. This is equivalent to having a cohomology ring isomorphic to that of $n(S^1 \times S^2)$. Hence, by Theorem 2(iii), the vanishing of all $\mu(i, j, k)$ is equivalent to $1 < \mathrm{cat}_{H_1}(M^3) \leqslant 3$. This finishes the proof of Theorem 4. $\square$

As an example let us consider the manifold M^3 obtained by surgery on a 2-component link L in a homology sphere. Let l be the linking number of the components and p_1/q_1, p_2/q_2 the surgery coefficients. Then, $H_1(M^3) \approx Z_{d_1} \oplus Z_{d/d_1}$ where $d = p_1 p_2 - l^2 q_1 q_2$ and $d_1 = \gcd(p_1, p_2, l)$ (we define $0/0$ as 0). If $|d| = 1$ then $\mathrm{cat}_{H_1}(M^3) = 1$; if $d = 0$ and $|d_1| = 1$ or if $d = d_1 = 0$ and L is a homology boundary link then $\mathrm{cat}_{H_1}(M^3) = 2$; if $d = d_1 = 0$ and L is not a homology boundary link then $\mathrm{cat}_{H_1}(M^3) = 3$; if $|d| > 1$ or $|d_1| > 1$ then $\mathrm{cat}_{H_1}(M^3) = 4$.

References

[1] S. Einlenberg and T. Ganea, On the Lusternik–Schnirelmann category of abstract groups, Ann. of Math. 65 (1957) 517–518.
[2] R.H. Fox, On the Lusternik–Schnirelmann category, Ann. of Math. 42 (1941) 333–370.
[3] T. Ganea, Sur quelques invariants numeriques du type d'homotopie, Cahiers de Topologie et Géometrie Differentielle IX (2) (1967) 181–241.
[4] J.C. Gómez-Larrañaga and F. González-Acuña, On the homological category of 3-manifolds, Rev. Mat. Univ. Complut. Madrid 4 (1991) 213–217.
[5] J.C. Gómez-Larrañaga and F. González-Acuña, The Lusternik–Schnirelmann category of 3-manifolds, Topology 31 (1992) 791–800.
[6] J. Hempel, 3-manifolds, Ann. of Math. Stud. 86 (Princeton University Press, 1976).
[7] H. Herrlich and G.E. Strecker, Category Theory (Allyn and Bacon Inc., Boston, 1973).
[8] E. Luft, Covering manifolds with open disks, Illinois J. Math. 13 (1969) 321–326.
[9] W. Magnus, Beziehungen zwischen Gruppen und Idealen in einem speziellen Ring, Math. Ann. III (1935) 259–280.
[10] J. Milnor, Link groups, Ann. of Math. (2) 59 (1954) 177–195.
[11] L. Montejano, A quick proof of Singhof's $\mathrm{cat}(M \times S^1) = \mathrm{cat}(M) + 1$ theorem, Manuscripta Math. 42 (1983) 49–52.
[12] R.P. Osborne and J.L. Stern, Covering manifolds with cells, Pacific J. Math. 30 (1969) 201–207.
[13] W. Singhof, Minimal coverings of manifolds with balls, Manuscripta Math. 29 (1979) 385–415.
[14] N. Smythe, Boundary links, in: Topology Seminar, Wisconsin, 1965, Ann. of Math. Stud. 60 (Princeton University Press, 1966) 69–72.
[15] J. Stallings, Homology and central series of groups, J. Algebra 2 (1965) 170–181.
[16] G.A. Swarup, On a theorem of C.B. Thomas, J. London Math. Soc. (2) 8 (1974) 13–21.
[17] C.B. Thomas, The oriented homotopy type of compact 3-manifolds, Proc. London Math. Soc. 19 (1967) 31–44.
[18] V.G. Turaev, Milnor invariants and Massey products, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov (POMI) 66 (1976) 189–203; English transl.: J. Soviet Math. 12 (1979) 128–137.
[19] V.G. Turaev, Nilpotent homotopy types of closed 3-manifolds, Topology, general and algebraic topology, and applications, in: Proc. Int. Conf., Leningrad, 1982, Lecture Notes in Math. 1060 (1984) 355–366.